Single-Particle Nondegeneracy and SU(3) Fermion Dynamical Symmetry

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Abstract

It is shown that the $SU(3)$ symmetry of the fermion dynamical symmetry model is essentially preserved even for highly nondegenerate spherical single-particle energies. The breaking of $SU(3)$ symmetry by single-particle energy terms for either normal deformation or superdeformation occurs only through an indirect Pauli effect and is significant only when the spherical single-particle splitting within shells is artificially large relative to that observed experimentally.

The shell model is commonly accepted as the microscopic basis for nuclear structure, but its practical implementation in medium and heavy nuclei requires a severe truncation of the model space. In the late 1980's, motivated by the phenomenological successes of the Interacting Boson Model [1] and building on the schematic fermion model of Ginocchio [2], we proposed a symmetry-dictated truncation scheme for the shell model termed the Fermion Dynamical Symmetry Model (FDSM) [3,4]. The symmetry limits of the model and perturbations around these limits have been explored extensively, and found to be consistent with a broad range of nuclear structure observations [5]. It is now of interest to examine in detail the excursions from the symmetry limits of the theory, in order to test its suitability as a systematic truncation procedure for quantitative shell model calculations in heavy nuclei. Although symmetry breaking has been investigated in some cases, there is as yet no systematic analysis of such terms in the theory. In this paper, we initiate such an analysis for the $Sp(6) \supset SU(3)$ limit of the FDSM.

The $Sp(6) \supset SU(3)$ dynamical symmetry of the FDSM may be identified with axially-symmetric rotational motion, and matrix elements derived in the symmetry limit or in perturbation around this limit have been shown to be in quantitative agreement with a variety of collective observables in heavy rotational nuclei [5]. The FDSM uses a modified Ginocchio coupling scheme that decomposes the single-particle angular momenta j of the shell model into an integer part k and a half-integer part i such that $\mathbf{j} = \mathbf{k} + \mathbf{i}$. For orbitals exhibiting an $Sp(6) \supset SU(3)$ dynamical symmetry, $k = 1$. In the lighter nuclei, there is a single value of i within a shell. In heavier nuclei there are typically 2–3 values of i within the normal-parity orbitals of a major shell, and for the enlarged valence spaces characteristic of superdeformation there may be as many as 5 values of i within a supershell [6]. A Hamiltonian with an $Sp(6) \supset SU(3)$ dynamical symmetry requires that single-particle energy terms corresponding to the same value of i be degenerate. Thus, the symmetry-limit Hamiltonian will generally exhibit a higher level of degeneracy than the realistic spherical single-particle shell model spectrum, and quantitative calculations must consider the effect on the symmetry-limit results of symmetry-breaking by single-particle energies. We emphasize that these remarks concern the splitting of the

single-particle spectrum for the *spherical* shell model. The additional splitting associated with quadrupole interactions (the algebraic analog of Nilsson splittings at finite deformation) is a separate issue that is handled in the FDSM through quadrupole–quadrupole coupling terms of the 2-body Hamiltonian. The major portion of these terms respects the symmetry [5], and the remainder may be incorporated numerically where needed [7].

In the discussion of the single-particle splitting, the question of the physically relevant scale for the phenomenon is important. As has been discussed extensively in Refs. [6,5,8], for collective properties of many-body systems a natural scale is set by the dominant correlation energies. The issue that must be addressed is not simply the size of the single-particle splitting, but its size relative to the correlation energy of the system and *how much of that splitting breaks the relevant symmetry.* In particular, no matter how large the single-particle splitting terms are, they will have no influence on the $SU(3)$ properties such as moments of inertia if they commute with the invariants of $SU(3)$. Furthermore, even if they do not commute, their influence will be greatly suppressed if correlation energies in the system produce large energy separations between irreducible representations of the dynamical symmetry.

To begin, we rewrite the single-particle energy in terms of the standard FDSM k–i basis:

$$\sum_j n_j e_j = \sum_{r,i} n_i^{(rr)0} e_i^r, \tag{1}$$

$$n_i^{(rr)0} = \sqrt{2\Omega_i}\left[b_{ki}^\dagger \tilde{b}_{ki}\right]^{(rr)0} \qquad e_i^r = \sum_j e_j \begin{bmatrix} k & i & j \\ k & i & j \\ r & r & 0 \end{bmatrix} \sqrt{\Omega_j/\Omega_i}. \tag{2}$$

with the square bracket denoting a normalized 9-j coefficient. The k–i basis $b_{ki}^\dagger$ has been defined in [2–4]; Ω_j and Ω_i are the pair degeneracies for the j shell and the shells associated with pseudospin i, respectively [$\Omega_j = j + \frac{1}{2}$, and $\Omega_i = (2k+1)(2i+1)/2$].

The states of the FDSM are classified according to a total heritage quantum number

u that measures the number of particles not coupled to coherent S and D pairs [4]. States below the first backbending region are dominantly $u = 0$ configurations. The mixing matrix elements associated with the splitting of the single-particle energies may be expressed as

$$\left\langle \lambda'\mu'u' \left| n_i^{(rr)0} \right| \lambda\mu u \right\rangle = \frac{\left\langle \lambda'\mu'u' \left| [n_i^{(rr)0}, C_{SU(3)}] \right| \lambda\mu u \right\rangle}{C(\lambda\mu) - C(\lambda'\mu')}. \tag{3}$$

where u and u' are the heritage quantum numbers and $C(\lambda\mu)$ is the usual eigenvalue of the quadratic $SU(3)$ Casimir operator $C_{SU(3)}$ evaluated in an SU_3 representation (λ, μ). Therefore, we must examine the commutation of the operators $n_i^{(rr)0}$ for $r = 0, 1, 2$ with the FDSM $SU(3)$ Casimir operator $C_{SU(3)}$. After some extensive algebra, one finds that $n_i^{(11)0}$ is the only component of the single-particle operator that can mix an $SU(3)$ irrep in the $u = 0$ bands with other $SU(3)$ irreps. The resulting mixing matrix element may be expressed in the form

$$\begin{aligned}
&\left\langle \lambda'\mu'u' \left| \sum_{r,i} n_i^{(rr)0} e_i^r \right| \lambda\mu u = 0 \right\rangle \\
&= \delta_{r1}\delta_{u'2} \left\{ -4 \sum_i e_i^1 \begin{bmatrix} i & i & 1 \\ i & i & 0 \\ 1 & 0 & 1 \end{bmatrix} \frac{\left\langle \lambda'\mu'u = 2 \left| [\mathcal{A}_{11}^\dagger(i)\tilde{\mathcal{A}}_{00}(i)]^{(11)0} \right| \lambda\mu u = 0 \right\rangle}{\Delta C} \right. \\
&\qquad \left. +4.472 \sum_i e_i^1 \begin{bmatrix} i & i & 1 \\ i & i & 0 \\ 1 & 0 & 1 \end{bmatrix} \frac{\left\langle \lambda'\mu'u = 2 \left| [\mathcal{A}_{11}^\dagger(i)\tilde{\mathcal{A}}_{20}(i)]^{(11)0} \right| \lambda\mu u = 0 \right\rangle}{\Delta C} \right\}
\end{aligned} \tag{4}$$

where $\Delta C \equiv C(\lambda\mu) - C(\lambda'\mu')$, and the pairing operators $\mathcal{A}$ are defined in Ref. [5]. Antisymmetrization requires that $K_1 + I_1$ and $K_2 + I_2$ be even integers; therefore, in Eq. (4) the allowed values for K_2 are 0 and 2 ($I_2 = 0$), and K_1 can only be 1 since $I_1 = 1$. This means that for the $r = 1$ term, $\sum_i n_i^{(11)0} e_i^1$ admixes the $u = 0$ and $u = 2$ irreps by changing a pair of particles in the symmetric $SU(3)$ representation $(\lambda, \mu) = (2, 0)$ into an antisymmetric representation $(0, 1)$ through the interaction $\mathcal{A}_{K_1 1}^\dagger(i)\tilde{\mathcal{A}}_{K_2 0}(i)$. No other

terms associated with the single-particle energies have any influence on the $SU(3)$ irreps of the $u = 0$ space.

The methods of Ref. [9] may be used to evaluate the matrix elements appearing in Eq. (4), and the preceding results may then be used to calculate numerically the influence of realistic spherical single-particle energies in the FDSM. Such work is in progress, but we now demonstrate that perturbation theory may be employed to obtain an immediate estimate for the limiting magnitude of $SU(3)$ symmetry-breaking caused by the single-particle splitting in the $u = 0$ representations. To second order in the perturbation, the difference in energy with and without splitting is

$$\Delta E = |E - E(u = 0)| = \frac{\Delta^2}{E(\lambda', \mu', u' = 2) - E(\lambda, \mu, u = 0)}, \tag{5}$$

where Δ is the mixing matrix element associated with the single-particle splitting [Eq. (4)]. As an upper limit, the matrix elements of the pairing operators $[\mathcal{A}^\dagger_{K_1 1}(i)\tilde{\mathcal{A}}_{K_2 0}(i)]^{(11)0}$ in Eq. (4) can be replaced by the diagonal matrix element of the monopole pairing in the $(n_1, 0)$ representation, which is $\frac{1}{4}n_i(2\Omega_i/3 - n_i + 2)$ (see Ref. [5]). The particle number n_i can be estimated as $(\Omega_i/\Omega_1)n_1$, where n_1 is the total number of particles in normal-parity levels and is around $2\Omega_1/3$ for the most deformed nuclei. Assuming that $(\lambda, \mu) = (n_1, 0)$ and $\Delta C = C(n_1 - 2, 1) - C(n_1, 0) = -3n_1$, the mixing matrix element Δ can be evaluated from Eq. (4) if we make an assumption concerning the relative phases of the two contributing terms. Let us take as extreme cases the assumption that the two matrix elements for $K_2 = 0$ and $K_2 = 2$ are either in phase (a result denoted by $\Delta_>$) or out of phase (denoted by $\Delta_<$):

$$\Delta_< = -\frac{0.0787}{\Omega_1}\sum_i e_i^1 \Omega_i \begin{bmatrix} i & i & 1 \\ i & i & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \Delta_> = -\frac{1.412}{\Omega_1}\sum_i e_i^1 \Omega_i \begin{bmatrix} i & i & 1 \\ i & i & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The excitation energy $|E(\lambda', \mu', u' = 2) - E(\lambda, \mu, u = 0)|$ appearing in Eq. (5) should be the energy required to break a pair (approximately the pairing gap energy), plus the excitation energy due to the change of the $SU(3)$ representation (which should be greater

than the bandhead energy of the γ and β bands). Thus its lower limit can be estimated as 2 MeV for normally deformed nuclei. The values of e_j and the corresponding e_i^r in the $k-i$ basis for the experimental single-particle splittings within a major shell are taken from [10]. From these quantities, the range of upper limits for ΔE versus the single-particle splitting can be estimated; the results are displayed in Fig. 1 for the 126–184 shell. The quantities $\Delta E_<$ and $\Delta E_>$ are the energy shifts calculated using $\Delta_<$ and $\Delta_>$, respectively. Thus, the shaded areas in these two figures represent the range of expected *upper limits* for the single-particle symmetry breaking. The horizontal axis $\rho = e_j/e_j(\exp)$ is a factor multiplying the spread of the single-particle (s. p.) energy scheme $\{e_j\}$ under consideration; hence $\rho = 1$ corresponds to the experimental s. p. splitting, and changing ρ corresponds to scaling the overall magnitude of the splitting. It is seen from Fig. 1 that the upper limit on the symmetry-breaking effect for the $SU(3)$ dynamical symmetry caused by the s. p. nondegeneracy is very small (from a fraction of an eV to several keV for the experimental spectra appropriate to the 126–184 shells). Only when the s. p. energy splitting is artifically large (say ten times larger than the splitting observed in a major shell) will the effect of $SU(3)$ dynamical symmetry breaking be significant and call into question the present perturbation theory analysis. We reiterate that the shaded region in Fig. 1 represents a range of estimated upper limits. The average symmetry-breaking in realistic situations may be even smaller than these estimates.

The small second-order energy shift justifies our use of perturbation theory for the present estimates, and implies that the symmetry-breaking admixture in the SU_3 wavefunction for $u = 0$ configurations is perturbative in size. Thus, the FDSM SU_3 wavefunction will remain essentially pure in the presence of the symmetry-breaking implied by a realistic spherical single-particle spectrum. This suggests that the single-particle symmetry-breaking terms will on average have only small influence on other observables such as transition rates and moments.

The preceding discussion has been formulated in terms of the FDSM for normal deformation, which assumes a single major shell of neutrons and protons as a valence

space, with effective interactions incorporating the influence of the truncation. An FDSM of superdeformations has been proposed [6] that employs a valence space of mainly two oscillator shells for neutrons and protons. An analysis similar to the present one may easily be carried out for such spaces, but we can obtain an immediate estimate of the influence of single-particle symmetry breaking for superdeformation from the present results. On the one hand, the summation over i in Eq. (4) should now run over two shells, which would increase the value of Δ by approximately a factor of two; on the other hand, the pair degeneracy Ω_1 of the shells responsible for the $SU(3)$ symmetry will also increase by about a factor of two in going to the superdeformed case. These two effects approximately cancel each other, keeping the value of Δ nearly the same for normal and superdeformation. However, the energy denominator in Eq. (5) is at least a factor of two larger for superdeformed configurations relative to the normally deformed case because of the enhanced collectivity. *Thus, the effect of the single-particle symmetry breaking on the energies is expected on general grounds to be even smaller for the superdeformed case than for the normally deformed case examined here.* This analysis implies that for superdeformed states, as for normally deformed states, the collective SU_3 wavefunction of the FDSM remains essentially pure in the presence of realistic single-particle energy splitting.

This pronounced stability of the $SU(3)$ dynamical symmetry for the FDSM is not a general property of fermion $SU(3)$ symmetries; it is a direct consequence of the particular structure of the Ginocchio S–D pairs from which the $SU(3)$ symmetry of the FDSM is realized [2,4,5]. *The FDSM SU*(3) *symmetry is defined in the pseudoorbital (*$k = 1$*) space*, with $U_k(3) \times U(\sum 2i + 1)$ as its higher symmetry ($U_k(3)$ is the unitary group associated with the k degree of freedom, $U(\sum 2i + 1)$ is the unitary group associated with the i degrees of freedom). Therefore all the operators in the k-space must commute with the $SU(3)$ invariants and there is no operator in the pseudospin space that can break the $SU(3)$ symmetry. The only way in which the s. p. terms can break the $SU(3)$ symmetry is through the higher symmetry $Sp(6)$ in which the $SU(3)$ symmetry is embedded, because the total wavefunction is required to be antisymmetric (i. e., the only allowed symmetry

breaking is indirect, through the Pauli effect). This constraint can force a change in the $SU(3)$ representation if the irrep of the i part of the wavefunction is changed. The only operator in the s. p. energy terms that can accomplish this is $n_i^{(11)0}$. In other fermion theories, such as the Elliott model [11] or the pseudo-$SU(3)$ model [12,13], the $SU(3)$ symmetry is embedded in a much larger group; therefore, there are many generators that could break directly the $SU(3)$ symmetry and one generally expects that the symmetry is more susceptible to symmetry breaking by single-particle energy terms.

It has been demonstrated in the Ginocchio model that for the vibrational symmetries of the FDSM, much of the effect of realistic single-particle splitting can be absorbed by a renormalization of parameters [14], leaving seniority as a reasonably good quantum number for low-lying vibrational states. We have shown here that single-particle symmetry breaking for the rotational symmetries of the FDSM has little influence on the corresponding symmetry. *Thus, the results of [14] and the present paper are strong evidence that spherical single-particle splitting leaves the collective aspects of all five dynamical symmetries of the* FDSM *largely intact, and we may expect on general grounds that collective properties obtained in the FDSM symmetry limits will survive the inclusion of realistic spherical single-particle spectra.*

To summarize, we have examined in this paper the effect of single-particle energy nondegeneracies on the FDSM $SU(3)$ dynamical symmetry for representations of zero heritage. We find that the symmetry is essentially preserved for any physically acceptable spectrum. Thus, the phenomenological successes of the FDSM $SU(3)$ model should survive in the limit of a realistic spherical single-particle energy spectrum for the collective aspects of both normal deformation and superdeformation. This result demonstrates that it is possible to construct classes of fermion $SU(3)$ symmetries that are virtually unperturbed even by large excursions in single-particle energies. It will be of interest to compare the present results with other fermion SU_3 symmetries such as the Elliott Model and the pseudo-SU_3 model, where one expects single-particle effects to have a larger influence on the symmetries. It will also be of interest to enquire whether the

present results are unique to the FDSM and its underlying Ginocchio coupling scheme, or whether there may exist additional classes of fermion symmetries having unusual stability with respect to single-particle energy splitting. Such theories have not been widely discussed in nuclear structure, but are of obvious interest for phenomena like identical bands that exhibit a particularly large stability of nuclear properties with respect to changing particle number.

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Fig. 1 Range of upper limits for the energy shift (relative to the symmetry limit) in $u = 0$ irreps caused by realistic single-particle energies in the 126–184 shell.